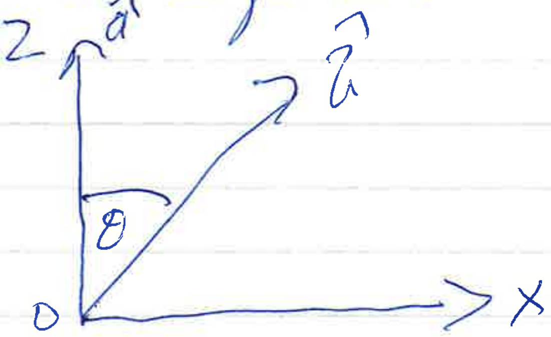
Consider $\mathrm{Prob}(\sigma_1 \cdot \hat{a} = 1 \,/\, \sigma_2 \cdot \vec{b} = 1)$

choose $\hat{a}$ along Z-axis, $\vec{b}$ in XY plane and take θ = angle between $\hat{a}$ and $\vec{b}$



Now $\mathrm{Prob}(\sigma_1 \cdot \hat{a} = 1 \,|\, \sigma_2 \cdot b = 1) = \dfrac{\mathrm{Prob}(\sigma_1 \cdot \hat{a} = 1 \;\&\; \sigma_2 \cdot b = 1)}{\mathrm{Prob}(\sigma_2 \cdot \vec{b} = 1)}$

Perturb Ψ_{singlet} by a unitary transformation $U(2)$ acting on particle ②.

So $\Psi_{\text{singlet}} \rightarrow \Psi' = \frac{1}{\sqrt{2}}\left(\alpha(1)\,U(2)\beta(2) - \beta(1)\,U(2)\alpha(2)\right)$

Write $U = \begin{pmatrix} 1-\varepsilon, & \eta' \\ \varepsilon', & 1-\eta \end{pmatrix}$ $\quad \varepsilon, \varepsilon', \eta, \eta'$ are four complex parameters.

and the unitarity relations

$$|1-\varepsilon|^2 + |\varepsilon'|^2 = 1$$
$$|1-\eta|^2 + |\eta'|^2 = 1$$
$$(1-\varepsilon^*)\,\eta' + \varepsilon'^*(1-\eta) = 0 .$$

Now $\mathrm{Prob}^{\Psi'}(\sigma_1 \cdot \hat{a} = 1 \;\&\; \sigma_2 \cdot \vec{b} = 1)$

$$= \left| \langle \alpha(1)| \cdot \left(\cos\theta/2 \,\langle \alpha(2)| + \sin\theta/2 \,\langle \beta(2)|\right) |\Psi'\rangle \right|^2$$

~~$= \langle \alpha(1)|\left(\cos\theta/2 \,\langle \alpha(2)| + \sin\theta/2 \,\langle \beta(2)|\right) \frac{1}{\sqrt{2}}\left(|\alpha(1)\rangle|\beta(\right.$~~

and $U(2)|\beta(2)\rangle = (1-\eta)|\beta(2)\rangle + \eta'|\alpha(2)\rangle$

$$\therefore \text{Prob}^{\Psi'}(\sigma_1\cdot\vec{a} = 1 \ \& \ \sigma_2\cdot\vec{b} = 1)$$

$$= \Big|\langle\alpha(1)|\left(\cos\theta/2\,\langle\alpha(2)| + \sin\theta/2\,\langle\beta(2)|\right) \frac{1}{\sqrt{2}}\Big(|\alpha(1)\rangle(1-\eta)|\beta(2)\rangle + \eta'|\alpha(1)\rangle|\alpha(2)\rangle - |\beta(1)\rangle(1-\varepsilon)|\alpha(2)\rangle - \varepsilon'|\beta(1)\rangle|\beta(2)\rangle\Big)\Big|^2$$

$$= \frac{1}{2}\Big|\left(\cos\theta/2\langle\alpha(1)|\langle\alpha(2)| + \sin\theta/2\,\langle\alpha(1)|\langle\beta(2)|\right) \cancel{\frac{1}{\sqrt{2}}}\Big((1-\eta)|\alpha(1)\rangle|\beta(2)\rangle + \eta'|\alpha(1)\rangle|\alpha(2)\rangle - (1-\varepsilon)|\beta(1)\rangle|\alpha(2)\rangle - \varepsilon'|\beta(1)\rangle|\beta(2)\rangle\Big)\Big|^2$$

$$= \Big|(1-\eta)\sin\theta/2 + \eta'\cos\theta/2\Big|^2$$

$$= \frac{1}{2}\Big(|1-\eta|^2\sin^2\theta/2 + |\eta'|^2\cos^2\theta/2 + 2\,\text{Re}\,(1-\eta^*)\eta'\sin\theta/2\cos\theta/2\Big)$$

$$= \frac{1}{2}\Big((1-|\eta'|^2)\sin^2\theta/2 + |\eta'|^2\cos^2\theta/2 + \text{Re}(1-\eta^*)\eta'\sin\theta\Big)$$

$$= \frac{1}{2}\Big(\sin^2\theta/2 + |\eta'|^2\cos\theta + \text{Re}(1-\eta^*)\eta'\sin\theta\Big)$$

$\leq \text{Prob}^{\Psi'}(\sigma_2\cdot\vec{b}=1 \;\&\; \sigma_1\cdot\vec{a}=+1)$
$+ \text{Prob}^{\Psi'}(\sigma_2\cdot\vec{b}=1 \;\&\; \sigma_1\cdot\vec{a}=-1)$

and $\text{Prob}^{\Psi'}(\sigma_2\cdot\vec{b}=1 \;\&\; \sigma_1\cdot\vec{a}=-1)$

$$= \left|\langle\beta(1)|\left(\cos\theta/2\,\langle\alpha(2)| + \sin\theta/2\,\langle\beta(2)|\right)|\Psi'\rangle\right|^2$$

$$= \frac{1}{2}\left|\left((1-\varepsilon)\cos\theta/2 + \varepsilon'\sin\theta/2\right)\right|^2$$

$$= \frac{1}{2}\left(\cos^2\theta/2 + |\varepsilon'|^2\cos\theta + \text{Re}\,(1-\varepsilon^*)\varepsilon'\sin\theta\right)$$

$*$

$$\therefore \text{Prob}^{\Psi'}(\sigma_1\cdot\vec{a}=1 \mid \sigma_2\cdot\vec{b}=1)$$

$$= \frac{\sin^2\theta/2 + |\eta'|^2\cos\theta + \text{Re}\,(1-\eta^*)\eta'\sin\theta}{1 + \cos\theta\left(|\eta'|^2 - |\varepsilon'|^2\right) + \sin\theta\cdot\text{Re}\left((1-\varepsilon^*)\varepsilon' + (1-\eta^*)\eta'\right)}$$

For invariant conditional probability we require: $* = \sin^2\theta/2$.

i.e. $|\eta'|^2\cos\theta + \text{Re}\,(1-\eta^*)\eta'\sin\theta$

$$= \cos\theta\sin^2\theta/2\left(|\eta'|^2 - |\varepsilon'|^2\right)$$

$**$ $$+ \sin\theta\sin^2\theta/2\;\text{Re}\left((1-\varepsilon^*)\varepsilon' + (1-\eta^*)\eta'\right)$$

always choose θ so that $**$ is
not satisfied.

Note that $**$ can be written as:

$$|\eta'|^2 \cos\theta \cdot \cos^2\theta/2 + \mathrm{Re}(1-\eta^{d})\eta' \sin\theta \cdot \cos^2\theta/2$$
$$+ \cos\theta \sin^2\theta/2 \cdot |\varepsilon'|^2 - \mathrm{Re}(1-\varepsilon^{d})\varepsilon' \sin\theta \sin^2\theta/2 = 0$$

or $$|\eta'|^2 \cos\theta(1+\cos\theta) + \mathrm{Re}(1-\eta^{d})\eta' \sin\theta(1+\cos\theta)$$
$$+|\varepsilon'|^2 \cos\theta(1-\cos\theta) - \mathrm{Re}(1-\varepsilon^{d})\varepsilon' \sin\theta(1-\cos\theta) = 0.$$

or $$\cos\theta\,\overset{a}{\left(|\eta'|^2+|\varepsilon'|^2\right)} + \cos^2\theta\,\overset{b}{\left(|\eta'|^2-|\varepsilon'|^2\right)}$$
$$+ \sin\theta\,\left\{\mathrm{Re}\left((1-\eta^{d})\overset{c}{\eta'} - (1-\varepsilon^{d})\varepsilon'\right)\right.$$
$$+ \sin\theta\cos\theta\,\mathrm{Re}\left((1-\eta^{d})\eta' \overset{d}{+} (1-\varepsilon^{d})\varepsilon'\right) = 0.$$

write $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$ $t = \tan\frac{1}{2}\theta$

The eq. becomes.
$$\frac{a(1-t^2)}{1+t^2} + b\frac{(1-t^2)^2}{(1+t^2)^2} + \frac{2ct}{1+t^2} + \frac{2dt(1-t^2)}{(1+t^2)^2} = 0$$

or. $$a(1-t^4) + b(1-t^2)^2 + 2ct(1+t^2) + 2dt(1-t^2)$$
or $$a - at^4 + b(1+t^4-2t^2) + 2ct + 2ct^3 + 2dt - 2dt^3$$
or $$t^4(b-a) + 2t^3(c-d) - 2bt^2 + 2t(c+d) + (a+b) = 0.$$

~~t^4~~

$$2t^4|\varepsilon'|^2 + 4t^3\,\mathrm{Re}\cdot(1-\varepsilon^*)\varepsilon'$$
$$+\,2t^2\left(|\eta'|^2 - |\varepsilon'|^2\right)$$
$$-\,4t\,\mathrm{Re}\,(1-\eta^2)\eta'$$
$$-\,2\,|\eta'|^2 = 0\,.$$

or.
$$t^4|\varepsilon'|^2 + 2t^3\,\mathrm{Re}\,(1-\varepsilon^*)\varepsilon'$$
$$+\,t^2\left(|\eta'|^2 - |\varepsilon'|^2\right)$$
$$-\,2t\,\mathrm{Re}\,(1-\eta^*)\eta'$$
$$-\,|\eta'|^2 = 0\,.$$

which is a quartic equation in $\tan\frac{1}{2}\theta$ and so has at most 4 roots in range $0 \leq \theta \leq \pi$.

now for $SU(2)$ take $\eta = \varepsilon^*$
$\eta' = -\varepsilon'^*$